ERRATA: TRANSFINITE LIMITS IN TOPOS THEORY

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Recall that in [TFL, Def. 3.6] a Grothendieck site $\mathbf{C}$ is called *admissible* if (1) it is coherent, (2) its topology is subcanonical, (3) for a finite family of objects $(C_i)_{i\in I}$ the coproduct $C = \coprod_{i\in I} C_i$ exists and $\{C_i \to C \mid i \in I\}$ is a covering, (4) coproducts are disjoint and stable under pullback.

Lemma 3.7 in [TFL] should be replaced by the following.

Lemma 1. *The following are equivalent for a coherent subcanonical site* $\mathbf{C}$*:*

(i) $\mathbf{C}$ *is admissible.*
(ii) *the essential image of the Yoneda functor*

$$\mathbf{y} : \mathbf{C} \to \mathrm{Sh}(\mathbf{C})$$

is closed under finite coproducts.

In particular for an admissible site $\mathbf{C}$ the Yoneda embedding $\mathbf{y} : \mathbf{C} \to \mathrm{Sh}(\mathbf{C})$ preserves coproducts.

We say that an object U of a site $\mathbf{C}$ is *weakly contractible* if any covering morphism $V \to U$ of the site splits. Above [TFL, Cor. 5.6] it is claimed that for $\mathbf{C}$ subcanonical an object U in $\mathbf{C}$ is weakly contractible if and only if the sheaf $\mathbf{y}(U)$ is weakly contractible in $\mathrm{Sh}(\mathbf{C})$. Here $\mathrm{Sh}(\mathbf{C})$ is endowed with the canonical topology, i.e. a covering morphism is the same as an epimorphism. It is not clear whether this is true in this form. This statement should be replaced by the following weaker one. The author would like to thank Catrin Mair for this observation.

Lemma 2. *The following are equivalent for an object* U *of an admissible site* $\mathbf{C}$*:*

(i) U *is weakly contractible in* $\mathbf{C}$,
(ii) $\mathbf{y}(U)$ *is weakly contractible in* $\mathrm{Sh}(\mathbf{C})$.

Proof. We only have to show (i) implies (ii). Consider an epimorphism of sheaves $F \to \mathbf{y}(U)$. Consider a family $\{\mathbf{y}(U_i) \to F \,|\, i \in I\}$ which is jointly epimorphic. The composite maps $\mathbf{y}(U_i) \to F \to \mathbf{y}(U)$ induce via the Yoneda Lemma a covering $\{U_i \to U \,|\, i \in I\}$. As U is quasi-compact there exists a subcovering $\{U_i \to U \,|\, i \in I'\}$ with $I' \subset I$ finite. Then by (i) there exists a splitting σ of the covering morphism $\coprod_{i\in I'} U_i \to U$. By the observation below Lemma 1 we have an isomorphism $\coprod_{i\in I'} \mathbf{y}(U_i) \xrightarrow{\sim} \mathbf{y}(\coprod_{i\in I'} U_i)$ in $\mathrm{Sh}(\mathbf{C})$. The composition of

$$\mathbf{y}(U) \xrightarrow{\sigma} \mathbf{y}(\coprod_{i\in I'} U_i) \cong \coprod_{i\in I'} \mathbf{y}(U_i) \to F$$

is a splitting of $F \to \mathbf{y}(U)$. □

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References

[TFL] Kerz, M. *Transfinite limits in topos theory*, Theory Appl. Categ. 31 (2016), No. **7**, 175–200.

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