THE GERSTEN CONJECTURE FOR MILNOR K-THEORY

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Abstract. We prove that the n-th Milnor K-group of an essentially smooth local ring over an infinite field coincides with the (n,n)-motivic cohomology of the ring. This implies Levine's generalized Bloch-Kato conjecture.

Contents

1. Introduction

The aim of this paper is to prove of a conjecture due to Alexander Beilinson [3] relating Milnor K-theory and motivic cohomology and to prove the Gersten conjecture for Milnor K-theory.

Theorem 1.1 (Beilinson's conjecture)**.** *For Voevodsky's motivic complexes of Zariski sheaves $\mathbb{Z}(n)$ [26] on the category of smooth schemes over an infinite field there is an isomorphism*

$$\mathcal{K}_n^M \xrightarrow{\sim} \mathcal{H}^n(\mathbb{Z}(n)) \tag{1}$$

for all $n \geq 0$.

Here $\mathcal{K}_*^M$ is the Zariski sheaf of Milnor K-groups (see Definition 2.1).
The surjectivity of the map in the theorem has been proven by Gabber [7] and Elbaz-Vincent/Müller-Stach [6], but only very little was known about injectivity at least if we are interested in torsion elements. Suslin/Yarosh proved the injectivity for discrete valuation rings of geometric type over an infinite field and $n = 3$ [27].

Date: 4/29/08.
The author is supported by *Studienstiftung des deutschen Volkes.*

We deduce Beilinson's conjecture from the Gersten conjecture for Milnor K-theory, i.e. the exactness of the Gersten complex

$$0 \longrightarrow \mathcal{K}_n^M|_X \longrightarrow \oplus_{x\in X^{(0)}} i_{x\,*}(K_n^M(x)) \longrightarrow \oplus_{x\in X^{(1)}} i_{x\,*}(K_{n-1}^M(x)) \longrightarrow \cdots$$

for a regular excellent scheme X over an infinite field. This can be done because the isomorphism (1) is known in the field case [20], [29] and there is an exact Gersten complex for motivic cohomology of smooth schemes. For general facts on the Gersten complex for cohomology theories the reader may consult [4].

As a consequence of Gersten's conjecture one deduces a Bloch formula relating Milnor K-theory and Chow groups

$$H^n(X, \mathcal{K}_n^M) = CH^n(X)$$

which was previously known only up to torsion [25] and for $n = 1, 2, \dim(X)$ due to Kato and Quillen [14], [23].

Furthermore one can deduce Levine's generalized Bloch-Kato conjecture for semi-local equicharacteristic rings [17] from the Bloch-Kato conjecture for fields, as well as the Milnor conjecture on quadratic forms over local rings.

Theorem 1.2 (Levine's Bloch-Kato conjecture). *Assume the Bloch-Kato conjecture. The norm residue homomorphism*

$$\chi_n : K_n^M(A)/l \longrightarrow H^n_{et}(A, \mu_l^{\otimes n})$$

is an isomorphism for $n > 0$ and all semi-local rings A containing a field k of characteristic not dividing l with $|k| = \infty$.

The proof of the Gersten conjecture is in a sense elementary and uses a mixture of methods due to Ofer Gabber, Andrei Suslin, and Manuel Ojanguren. There are two new ingredients:

In Section 3 we construct a co-Cartesian square motivated by motivic cohomology which was suggested to hold by Gabber [7].

Section 4 extends the Milnor-Bass-Tate sequence [2], [19] to semi-local rings. This provides norm maps on Milnor K-groups for finite, étale extensions of semi-local rings which are constructed in Section 5. The existence of these generalizations was conjectured by Kahn [11], Elbaz-Vincent and Müller-Stach.

In Section 6 our main theorem is proved namely:

Theorem 1.3. *Let A be a regular connected semi-local ring containing a field with quotient field F. Assume that each residue field of A is infinite. Then the map*

$$i_n : K_n^M(A) \longrightarrow K_n^M(F)$$

is universally injective for all $n \geq 0$.

Applications of this theorem, in particular Theorem 1.1 and Theorem 1.2, are discussed in Section 7.

The strategy of our proof of the main theorem is as follows:

First we reduce the proof to the case in which A is defined over an infinite perfect field k and A is the semi-local ring associated to a collection of closed points of an affine, smooth variety X/k. This reduction is accomplished by a Néron-Popescu desingularization [28]

and using the norms constructed in Section 5. Then we apply induction on $d = \dim(A)$ for all n at once.
By the co-Cartesian square and Gabber's geometric presentation theorem one can assume $X = \mathbb{A}^d_k$.
Using the generalized Milnor-Bass-Tate sequence and the induction assumption that injectivity is already proved for rings of lower dimension one gets injectivity in dimension d.
Gabber used a similar mechanism to prove the surjectivity of the map (1) in [7]. His proof as well as the proof of Elbaz-Vincent/Müller-Stach for this statement can be simplified using the methods developed in Section 4, compare [15], [16].

2. Milnor K-Theory

In this section we recall the definition of Milnor K-Theory of semi-local rings and some properties needed later – following [20] and [27].
Let A be a unital commutative ring, $T(A^*)$ the $\mathbb{Z}$-tensor algebra over the units of A. Let I be the homogeneous ideal in $T(A^*)$ generated by elements $a\otimes(1-a)$ with $a, 1-a \in A^*$. Elements of I are usually called Steinberg relations.

Definition 2.1. *With the above notation we define the Milnor K-ring of A to be $K^M_*(A) = T(A^*)/I$.*

By $\mathcal{K}^M_*$ we denote the associated Zariski sheaf of the presheaf $U \mapsto K^M_*(\Gamma(U))$ on the category of schemes.
The residue class of an element $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in $K^M_n(A)$ is denoted $\{a_1, a_2, \ldots, a_n\}$.
In what follows we will be concerned with the Milnor ring of a localization of a semi-local ring with sufficiently many elements in the residue fields. Sufficiently many will always depend on the context. Although results are usually discussed only for infinite residue fields, an argument in Section 6 and 7 uses Milnor K-groups of semi-local rings with finite residue fields.
The next lemma is a generalization of [20, Lemma 3.2].

Lemma 2.2. *Let A be a semi-local ring with infinite residue fields and B a localization of A. For $a, a_1, a_2 \in B^*$ we have*

$$\{a, -a\} = 0$$

and

$$\{a_1, a_2\} = -\{a_2, a_1\}\ .$$

For the proof we misuse notation and write elements of A and the associated induced elements in B by the same symbols.

Proof. For simplicity we discuss only the case A local. It is clear that the second relation follows from the first since

$$\{a_1, a_2\} + \{a_2, a_1\} = \{a_1a_2, -a_1a_2\} - \{a_2, -a_2\} - \{a_1, -a_1\}\ . \tag{2}$$

The proof of the relation $\{a, -a\} = 0 \in K^M_2(B)$ for $a \in A^*$, understood to mean the element induced in B^*, goes as follows. If $1 - a \in A^*$ write

$$-a = \frac{1-a}{1-1/a} \tag{3}$$

so that

$$\{a,-a\} = \{a, 1-a\} - \{a, 1-a^{-1}\} = 0 \;.$$

If $1-a \notin A^*$ but $a \in A^*$, notice that for $s \in A^*$, $\bar{s} \neq 1$ we have $1 - as \in A^*$ so that

$$\begin{aligned} 0 = \{as, -as\} &= \{a,-a\} + \{s,-s\} + \{a,s\} + \{s,a\} \\ &= \{a,-a\} + \{a,s\} + \{s,a\} \;. \end{aligned}$$

So if we choose $s_1, s_2 \in A^*$ with $\bar{s}_1 \neq 1 \neq \bar{s}_2$ and $\bar{s}_1\bar{s}_2 \neq 1$ we get from the last equations

$$\begin{aligned} \{a,-a\} = -\{a, s_1s_2\} - \{s_1s_2, a\} &= -\{a,s_1\} - \{s_1,a\} - \{a,s_2\} - \{s_2,a\} \\ &= \{a,-a\} + \{a,-a\} \;. \end{aligned}$$

Suppose now $a \in A$, $a \in B^*$ but $a \notin A^*$. Then $1-a \in A^*$ and $1-a^{-1} \in B^*$. So we can write $-a$ as in (2). which again gives

$$\{a,-a\} = \{a, 1-a\} - \{a, 1-a^{-1}\} = 0 \;.$$

In the general case let $a = b/c$ for $b, c \in A$ and $b, c \in B^*$

$$\{a,-a\} = \{b/c, -b/c\} = \{b,-b\} + \{c,c\} - \{c,-b\} - \{b,c\} \;.$$

What we have already proved together with (2) gives $\{c,c\} = \{c, (-1)(-c)\} = \{c,-1\}$ and

$$\{a,-a\} = \{c,-1\} - \{c,-b\} + \{c,b\} = 0 \;.$$

The general case of the lemma, i.e. for A semi-local, follows similarly. One only has to make a more careful choice of the element s. □

Let as before A be a semi-local ring with infinite residue fields.

Proposition 2.3. *Let $a_1, \ldots, a_n$ be in A^* such that $a_1 + \cdots + a_n = 1$, then*

$$\{a_1, \ldots, a_n\} = 0 \in K_n^M(A) \;.$$

Proof. If the reader is interested she can find a proof in [27, Corollary 1.7]. □

Later we will need another simple lemma. Let B be a localization of a semi-local ring.

Lemma 2.4. *For $a_1, a_2, a_1 + a_2 \in B^*$ we have*

$$\{a_1, a_2\} = \{a_1 + a_2, -\frac{a_2}{a_1}\} \;.$$

Proof. We have

$$\begin{aligned} 0 &= \{\frac{a_1}{a_1+a_2}, \frac{a_2}{a_1+a_2}\} \\ &= \{a_1,a_2\} - \{a_1, a_1+a_2\} - \{a_1+a_2, a_2\} + \{a_1+a_2, a_1+a_2\} \\ &= \{a_1,a_2\} + \{a_1+a_2, a_1\} - \{a_1+a_2, a_2\} + \{a_1+a_2, -1\} \\ &= \{a_1,a_2\} - \{a_1+a_2, -\frac{a_2}{a_1}\} \;. \end{aligned}$$

The first equation is the standard Steinberg relation, the third equation comes from the relations of Lemma 2.2. □

Remark 2.5. *We do not know whether Proposition 2.3 holds in case $a_1, \ldots, a_n$ are elements in B^* with $a_1 + \cdots + a_n = 1$.*

3. A co-Cartesian square

The theorem we prove in this section was suggested to hold by Gabber [7]. In order to motivate it consider the following geometric data:
Let $f : X' \to X$ be an étale morphism of smooth varieties and $Z \subset X$ a closed subvariety such that $f^{-1}(Z) \to Z$ is an isomorphism. Let $U' = X' - f^{-1}(Z)$ and $U = X - Z$. Then in the derived category of mixed motives DM_{gm}^{eff} over a perfect field [31] there is a distinguished triangle of the form

$$M_{gm}(U') \longrightarrow M_{gm}(X') \oplus M_{gm}(U) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}(U')[1] \,.$$

This can be easily deduced from [30, Proposition 5.18].
Therefore in case X is semi-local the sequence

$$H^n(X,\mathbb{Z}(n)) \longrightarrow H^n(X',\mathbb{Z}(n)) \oplus H^n(U,\mathbb{Z}(n)) \longrightarrow H^n(U',\mathbb{Z}(n)) \longrightarrow 0 \tag{4}$$

is exact, because $H^{n+1}(X,\mathbb{Z}(n)) = 0$ as X is semi-local. In fact the Zariski sheaf $\mathbb{Z}(n)$ vanishes in degrees greater than n so that the vanishing of $H^{n+1}(X,\mathbb{Z}(n))$ follows from the spectral sequence

$$\mathbb{H}^l_{Zar}(X,\mathcal{H}^k(\mathbb{Z}(n))) \Longrightarrow H^{l+k}(X,\mathbb{Z}(n))$$

and [30, Lemma 4.28].
Let $A \subset A'$ be a local extension of factorial semi-local rings with infinite residue fields, i.e. the morphism $Spec(A') \to Spec(A)$ is dominant, maps closed points to closed points and is surjective on the latter. Let $f, f_1 \neq 0$ be in A such that $f_1|f$ and $A/(f) \cong A'/(f)$. Denote the localization of A with respect to $\{1, f, f^2, \ldots\}$ resp. $\{1, f_1, f_1^2, \ldots\}$ by A_f resp. A_{f_1}.
As according to the Beilinson conjectures the n-th Milnor K-group of a reasonably good ring – for example a localizations of a smooth local rings – should coincides with its (n,n)-motivic cohomology the exact sequence (4) motivates:

Theorem 3.1. *The diagram*

$$\begin{array}{ccc} K_n^M(A_{f_1}) & \longrightarrow & K_n^M(A_f) \\ \downarrow & & \downarrow \\ K_n^M(A'_{f_1}) & \longrightarrow & K_n^M(A'_f) \end{array}$$

is co-Cartesian.

Before we give the slightly tedious proof the reader should remember that for any $n > 0$ the unit group A'^* is generated by A^* and $1 + f^n A'$. This is easily shown by induction on n.

Proof. For simplicity we restrict to the case A, A' local. Let $\pi \in A$ be an irreducible factor of f/f_1, $f = \pi f'$, and B resp. B' the localization $A_{f'}$ resp. $A'_{f'}$. By induction it is clearly sufficient to show

$$\begin{array}{ccc} K_n^M(B) & \longrightarrow & K_n^M(B_\pi) \\ \downarrow & & \downarrow \\ K_n^M(B') & \longrightarrow & K_n^M(B'_\pi) \end{array} \tag{5}$$

is co-Cartesian.
In order to see this one has to construct a multilinear map

$$\lambda : ((B'_\pi)^*)^{\times n} \longrightarrow K_n^M(B') \oplus K_n^M(B_\pi)/K_n^M(B)$$

which induces an isomorphism compatible with (5)

$$K_n^M(B'_\pi) \cong K_n^M(B') \oplus K_n^M(B_\pi)/K_n^M(B) \; .$$

Because $B'^* = B^*(1+\pi A')$ one can write each element of an n-tuple

$$(a_1, \dots, a_n) \in (B'_\pi)^{\times n}$$

as

$$a_i = \pi^{j_i} y_i (1 + \pi x_i) \tag{6}$$

$i = 1, \dots, n$, with $j_i \in \mathbb{Z}$, $y_i \in B^*$ and $x_i \in A'^*$. The element x_i can be assumed to be invertible in A' since if it was not invertible one could write

$$y_i(1+\pi x_i) = y_i/(1+\pi)\,[1 + \pi(1 + x_i + \pi x_i)] \; .$$

Now we translate some results from [27] into our setting. Define the multiplicative group $A'^*_{(1)}$ as $1+\pi A'$, the set $A'^*_{(inv)}$ as $1+\pi A'^*$ and the map

$$\rho : A'^*_{(inv)} \times ((B'_\pi)^*)^{\times n-1} \longrightarrow K_n^M(B')$$

by

$$\rho((1+\pi x), \pi^{j_2} w_2, \dots, \pi^{j_n} w_n) = \{(1+\pi x), \frac{w_2}{(-x)^{j_2}}, \dots, \frac{w_n}{(-x)^{j_n}}\}$$

for $(w_i, \pi) = 1$, $i = 2, \dots, n$.
Now let U be the union of $(A'^*_{(1)}) \times ((B'_\pi)^*)^{\times n-1}$, $(B'_\pi)^* \times (A'^*_{(1)}) \times ((B'_\pi)^*)^{\times n-2}$ etc.

Lemma 3.2. *The map ρ extends uniquely to a well defined skew-symmetric multilinear map*

$$U \longrightarrow K_n^M(B') \; .$$

Proof. From Sublemma 3.3 we deduce that we can extend ρ to a canonical multilinear map from its original domain of definition $(A'^*_{(inv)}) \times ((B'_\pi)^*)^{\times n-1}$ to the domain $(A'^*_{(1)}) \times ((B'_\pi)^*)^{\times n-1}$.

Sublemma 3.3. *For $1+\pi x = (1+\pi x_1)(1+\pi x_2)$ and $x, x_1 \in A'^*$, $x_2 \in A'$ with $x_2 \in (B')^*$*

$$\{1+\pi x, 1/(-x)\} = \{1+\pi x_1, 1/(-x_1)\} + \{1+\pi x_2, 1/(-x_2)\} \; .$$

Proof. For sake of completeness we recall the proof from [27, Lemma 3.5]. Let

$$\eta = \{1+\pi x, -x\} - \{1+\pi x_1, -x_1\} - \{1+\pi x_2, -x_2\} \, .$$

We have

$$\begin{aligned} \eta &= \{1+\pi x_1, \frac{x}{x_1}\} + \{1+\pi x_2, \frac{x}{x_2}\} \\ &= \{-\frac{x_1}{x_2}, \frac{x}{x_1}\} + \{-\frac{x_2}{x_1}, \frac{x}{x_2}\} \\ &= \{-\frac{x_1}{x_2}, x\} + \{x_2, x_1\} + \{-\frac{x_2}{x_1}, x\} + \{x_1, x_2\} \\ &= 0 \end{aligned}$$

where the second equation follows from

$$\begin{aligned} \frac{x}{x_1} &= 1 + \frac{x_2}{x_1}(1+\pi x_1) \\ \frac{x}{x_2} &= 1 + \frac{x_1}{x_2}(1+\pi x_2) \, . \end{aligned}$$

□

Next we have to check what happens if there are two entries of $A'^*_{(1)}$ in an n-tupel. The next sublemma shows that the definition of ρ does not depend on how we eliminate the factors of π from our n-tupel by using either of the two distinguished $A'^*_{(1)}$ entries.

Sublemma 3.4. *For $x_1, x_2 \in A'^*$ one has*

$$\{1-\pi x_1, 1-\pi x_2, \frac{1}{x_1}\} = \{1-\pi x_1, 1-\pi x_2, \frac{1}{x_2}\}$$

Proof. Because of Proposition 2.3 we have

$$\begin{aligned} \{1-\pi x_1, 1-\pi x_2, \frac{x_2}{x_1}\} &= \{-\frac{x_2}{x_1}(1-\pi x_1), 1-\pi x_2, \frac{x_2}{x_1}\} \\ &= 0 \end{aligned}$$

□

□

As we saw above $((B'_\pi)^*)^{\otimes n}$ is generated by U and $V = ((B_\pi)^*)^{\otimes n}$. So one defines λ on U by ρ and on V by the natural surjection $V \to K_n^M(B_\pi)$.
It is immediately clear that λ does not depend on the factorization (6) or what is the same on the special decomposition of an element of $((B'_\pi)^*)^{\otimes n}$ into elements of U and V.
It is more difficult to show that λ maps the Steinberg relations to zero. Denote by Λ the subgroup of $((B'_\pi)^*)^{\otimes n}$ generated by elements of the form $a_1 \otimes \cdots \otimes a_n$ with $a_i + a_j = 1$ for some $i \neq j$. We have to show $\lambda(\Lambda) = 0$.

Lemma 3.5. *The group Λ is generated by elements of the form*

(i) $a_1 \otimes \cdots \otimes a_n$ *with* $a_1, \ldots, a_n \in (B'_\pi)^*$ *and* $a_i + a_j = 0$ *for some* $i \neq j$.
(ii) $a \otimes (1-a) \otimes a_3 \otimes \cdots \otimes a_n$ *with* $a, 1-a, a_3, \ldots, a_n \in (B_\pi)^*$
(iii) $a\pi \otimes (1-a\pi) \otimes a_3 \otimes \cdots \otimes a_n$ *with* $a \in A'$, $a \in B'^*$ *and* $a_i \in (B'_\pi)^*$ *for* $i = 3, \ldots, n$.

(iv) $a\pi^i \otimes (1 - a\pi^i) \otimes (1 - f^\infty x) \otimes a_4 \otimes \cdots \otimes a_n$ *with* $i \geq 0$, $a, 1 - a\,\pi^i \in B'^*$, $a_4, \dots, a_n \in (B'_\pi)^*$ *and* $x \in A'^*$.

Here f^∞ means an arbitrarily high power of f. For later use in part (iv) it has to be chosen so large that equation (9) below becomes true, for example such that $f^\infty\,\pi^{-i}\,a^{-1} \in A'^*$.

Proof. We have to recall the five-term relation whose proof is an elementary but tedious symbolic argument which is left to the reader.

Sublemma 3.6 (Five-term relation)**.** *With*

$$[a] = a \otimes (1-a) \in (B'_\pi)^* \otimes (B'_\pi)^*$$

we have

$$\begin{aligned}(7) \qquad [x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)] = \\ x \otimes (1-x)/(1-y) + (1-x)/(1-y) \otimes x\end{aligned}$$

if $x, y, 1-x, 1-y, x-y \in (B'_\pi)^*$.

For the proof of Lemma 3.5 we use induction on n. For $n = 2$ we will use the five-term relation.
$\mathbf{n = 2}$: Let $a, 1-a \in (B'_\pi)^*$. We will express $[a]$ in terms of relations (i)-(iii). Choose $x' \in A'^*$ such that $y = (1 + f^\infty x')a \in B_\pi$ and let x be $1 + f^\infty x'$. Here f^∞ is such a large power of f that the following arguments work, depending only on a.
The five-term relation for x and y as just defined gives (modulo the relations (i) which covers the right side of (7)) $[y/x] = [a]$ in terms of $[y]$ which is covered by (ii) and $[x], [(1-x)/(1-y)], [(1-x^{-1})/(1-y^{-1})]$ which are covered by (iii) as we will show now. The latter elements are of the form $[1 + \pi^i a]$ with $a \in A'$ and $a \in B'^*$, $i > 0$. We will see by induction on i that we can suppose $i = 1$. Set $x = 1 + \pi$ and $y = (1+\pi)(1+\pi^i a)$. If we again use the five-term relation with our new definition of x and y in equation (7) we get the result.
$\mathbf{n = 3}$:
Modulo relation (i) we have to show that an element $a \otimes (1-a) \otimes b$ with $a, 1-a, b \in (B'_\pi)^*$ can be expressed in terms of relations (ii)-(iv). According to what we proved for the case $n = 2$ we can assume either $a, 1-a \in (B_\pi)^*$ or $a/\pi, 1-a \in A'$ and $\in B'^*$ without denominators. The latter case is comprehended by (iii), the former by (ii) and (iv) if we factor b in the form $(B'_\pi)^* = (1 - A'^*\,f^\infty) \cdot (B_\pi)^*$.
$\mathbf{n > 3}$:
This is simple if we proceed in analogy to case $n = 3$. □

Compatibility of λ with (i): Assume without loss of generality $n = 2$. Given an element $\pi^i a(1-\pi x) \otimes -\pi^i a(1-\pi x) \in (B'_\pi)^* \otimes (B'_\pi)^*$ with $x \in A'^*$ and $a \in B^*$ we get

$$\begin{aligned}\lambda(\pi^i a(1-\pi x) \otimes -\pi^i a(1-\pi x)) &= [\{1-\pi x, -\frac{a}{x^i}\} + \{1-\pi x, 1-\pi x\} \\ &\quad + \{\frac{a}{x^i}, 1-\pi x\}] \oplus \{\pi^i a, -\pi^i a\} \\ &= 0 \oplus 0\,.\end{aligned}$$

Compatibility of λ with (ii): Clear.

Compatibility of λ with (iii): This follows from Sublemma 3.3 because we have

$$\lambda(a\pi \otimes (1-a\pi) \otimes a_3 \otimes \cdots \otimes a_n) = \{a/a, 1-a\pi, \ldots\} \oplus 0 = 0 \oplus 0 \,.$$

Compatibility of λ with (iv): If $i=0$ this is trivial, therefore assume $i>0$. Let $a = a_1/a_2$ be an irreducible fraction with $a_1, a_2 \in A'$, $a_1, a_2 \in B'^*$ and $1-a_2 \in A'^*$. Write further

$$1 - f^\infty x = \frac{(1-\pi^i a_1)(1-f^\infty x)}{1-\pi^i a_1} = \frac{1-\pi^i[a_1 + x f^\infty \pi^{-i}(1-\pi^i a_1)]}{1-\pi^i a_1}$$

So it is sufficient to show

$$\zeta := \lambda(a\pi^i \otimes (1-a\pi^i) \otimes (1-\pi^i a_1) \otimes a_4 \otimes \cdots \otimes a_n) = 0 \tag{8}$$

$$\lambda(a\pi^i \otimes (1-a\pi^i) \otimes (1-\pi^i[a_1 + x f^\infty \pi^{-i}(1-\pi^i a_1)]) \otimes a_4 \otimes \cdots \otimes a_n) = 0 \tag{9}$$

The demonstration of (9) is almost identical to that of (8), so we restrict to (8).
We know from the compatibility of λ with (iii) and the proof of Lemma 3.5 that

$$\lambda(a_1\pi^i \otimes (1-a_1\pi^i)) = 0 \,.$$

This gives the first equality in

$$\begin{aligned}
\zeta &= \{\frac{1}{a_2}, 1 - \frac{a_1}{a_2}\pi^i, 1 - a_1\pi^i\} \oplus 0 = \{\frac{1}{a_2}, -a_2(1-\frac{a_1}{a_2}\pi^i), 1-a_1\pi^i\} \oplus 0 \\
&= \{\frac{1}{a_2}, a_1\pi^i - a_2, 1 - a_1\pi^i\} \oplus 0 \\
&= \{\frac{1}{a_2}, 1-a_2, -\frac{1-a_1\pi^i}{a_1\pi^i - a_2}\} \oplus 0 = 0 \,.
\end{aligned}$$

The fourth equality follows from Lemma 2.4.
This finishes the proof of Theorem 3.1 as the reader checks without difficulties that λ is an inverse to

$$K_n^M(B') \oplus K_n^M(B_\pi)/K_n^M(B) \longrightarrow K_n^M(B'_\pi) \,.$$

□

4. Milnor-Bass-Tate sequence

The most fundamental result in Milnor K-theory of fields is the short exact sequence due to Milnor, Bass and Tate [2], [19]

$$0 \longrightarrow K_n^M(F) \longrightarrow K_n^M(F(t)) \longrightarrow \oplus_\pi K_{n-1}^M(F[t]/(\pi)) \longrightarrow 0 \tag{10}$$

where F is a field and the direct sum is over all irreducible, monic $\pi \in F[t]$.
It calculates Milnor K-groups of the function field of a projective line.
In order to prove Beilinson's conjecture we generalize this sequence to the realm of local rings. Let A be a semi-local domain with infinite residue fields, F its quotient field. Furthermore we assume A to be factorial in order to simplify our notation. For a description of the general case, which is not needed in the proof of our main theorem, compare Section 5.
For a local ring version of (10) one has to replace the group $K_n^M(F(t))$ by a group of symbols in general position denoted $K_n^t(A)$.

Definition 4.1. *An n-tuple of rational functions*

$$(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_n}{q_n}) \in F(t)^n$$

with $p_i, q_i \in A[t]$ and p_i/q_i a reduced fraction for $i = 1, \dots n$ is called feasible if the highest nonvanishing coefficients of p_i, q_i are invertible in A and for irreducible factors u of p_i or q_i and v of p_j or q_j ($i \neq j$), $u = av$ with $a \in A^*$ or $(u, v) = 1$.

Before coming to the definition of $K_n^t(A)$ we have to replace ordinary tensor product.

Definition 4.2. *Define*

$$\mathcal{T}_n^t(A) = \mathbb{Z}\langle\{(p_1, \dots, p_n) | (p_1, \dots, p_n) \textit{ feasible, } p_i \in A[t] \textit{ irreducible or unit}\}\rangle / Linear$$

Here *Linear* denotes the subgroup generated by elements

$$(p_1, \dots, ap_i, \dots, p_n) - (p_1, \dots, a, \dots, p_n) - (p_1, \dots, p_i, \dots, p_n)$$

with $a \in A^*$.
By bilinear factorization the element

$$(p_1, \dots, p_n) \in \mathcal{T}_n^t(A)$$

is defined for every feasible n-tuple with $p_i \in F(t)$.
Now define the subgroup $St \subset \mathcal{T}_n^t(A)$ to be generated by feasible n-tuples

$$(p_1, \dots, p, 1-p, \dots, p_n) \tag{11}$$

and

$$(p_1, \dots, p, -p, \dots, p_n) \tag{12}$$

with $p_i, p \in F(t)$.

Definition 4.3. *Define*

$$K_n^t(A) = \mathcal{T}_n^t(A)/St$$

We denote the image of $(p_1, \dots, p_n)$ in $K_n^t(A)$ by $\{p_1, \dots, p_n\}$.
Now the main theorem of this section reads:

Theorem 4.4. *There exists a split exact sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^t(A) \longrightarrow \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \longrightarrow 0 \tag{13}$$

where the direct sum is over all monic, irreducible $\pi \in A[t]$.

The first map in sequence (13) is induced by the inclusion $A \to F(t)$. The second is a generalization of the tame symbol whose construction will be given below.
In the proof of the Gersten conjecture we need a slightly refined version of this theorem. Let $0 \neq p \in A[t]$ be an arbitrary monic polynomial. Define the group $K_n^t(A, p)$ in analogy to $K_n^t(A)$ but this time a tuple

$$(p_1/q_1, p_2/q_2, \dots, p_n/q_n)$$

is feasible if additionally all p_i, q_i are coprime to p. The proof of the following theorem is almost identical to the proof of Theorem 4.4.

Theorem 4.5. *The sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^t(A,p) \longrightarrow \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \longrightarrow 0$$

is split exact where the direct sum is over all $\pi \in A[t]$ monic and irreducible with $(\pi, p) = 1$.

Lemma 4.6. *For every feasible n-tuple $(p_1, \dots, p_n)$ and $1 \leq i < n$ we have*

$$\{p_1, \dots, p_i, p_{i+1}, \dots, p_n\} = -\{p_1, \dots, p_{i+1}, p_i, \dots, p_n\} \in K_n^t(A) \ .$$

Proof. We can suppose $n = 2$ and $p_1, p_2 \in A[t]$ irreducible or units, then

$$\{p_1, p_2\} + \{p_2, p_1\} = \{p_1 p_2, -p_1 p_2\} - \{p_2, -p_2\} - \{p_1, -p_1\} = 0 \ .$$

$\square$

Proof of Theorem 4.4.

Step 1: The homomorphism $i_n : K_n^M(A) \to K_n^t(A)$ is injective.

We construct a left inverse ψ_n of i_n by associating to a polynomial its highest coefficient (specialization at infinity). This gives a well defined map $\psi_n : \mathcal{T}_n^t(A) \to K_n^M(A)$. We have to show ψ_n factors through the Steinberg relations. As concerns relation (12) one gets

$$\psi_n((p_1, \dots, p, -p, \dots, p_n)) = \{\psi_1(p_1), \dots, \psi_1(p), -\psi_1(p), \dots, \psi_1(p_n)\} = 0 \ .$$

For relation (11) one has to distinguish several cases. Given $p, q \in A[t]$, $\deg(p) > \deg(q)$ we have

$$\begin{aligned}\psi_n((p_1, \dots, p/q, 1 - p/q, \dots, p_n)) &= \psi_n((p_1, \dots, p/q, (q-p)/q, \dots, p_n)) \\ &= \{\psi_1(p_1), \dots, \psi_1(p)/\psi_1(q), -\psi_1(p)/\psi_1(q), \dots, \psi_1(p_n)\} = 0\end{aligned}$$

for $\deg(p) < \deg(q)$

$$\begin{aligned}\psi_n((p_1, \dots, p/q, 1 - p/q, \dots, p_n)) &= \psi_n((p_1, \dots, p/q, (q-p)/q, \dots, p_n)) \\ &= \{\psi_1(p_1), \dots, \psi_1(p)/\psi_1(q), 1, \dots, \psi_1(p_n)\} = 0\end{aligned}$$

for $\deg(p) = \deg(q) = \deg(q - p)$

$$\begin{aligned}\psi_n((p_1, \dots, p/q, 1 - p/q, \dots, p_n)) &= \psi_n((p_1, \dots, p/q, (q-p)/q, \dots, p_n)) \\ &= \{\psi_1(p_1), \dots, \psi_1(p)/\psi_1(q), 1 - \psi_1(p)/\psi_1(q), \dots, \psi_1(p_n)\} = 0\end{aligned}$$

for $\deg(q) = \deg(p) > \deg(p - q)$

$$\begin{aligned}\psi_n((p_1, \dots, p/q, 1 - p/q, \dots, p_n)) &= \psi_n((p_1, \dots, p/q, (q-p)/q, \dots, p_n)) \\ &= \{\psi_1(p_1), \dots, 1, \psi_1(q-p)/\psi_1(q), \dots, \psi_1(p_n)\} = 0 \ .\end{aligned}$$

Therefore $\psi_n : K_n^t(A) \to K_n^M(A)$ is well defined and $\psi_n \circ i_n = id$.

Step 2: Constructing the homomorphisms $K_n^t(A) \to K_{n-1}^M(A[t]/(\pi))$.

Let $\pi \in A[t]$ be a monic irreducible. For every such π one constructs group homomorphisms

$$\partial_\pi : K_n^t(A) \longrightarrow K_{n-1}^M(A[t]/(\pi))$$

such that

$$\partial_\pi(\{\pi, p_2, \dots, p_n\}) = \{\bar{p}_2, \dots, \bar{p}_n\}$$

for $p_i \in A[t]$ and $(p_i, \pi) = 1$, $i = 2, \dots, n$. Clearly the last equation characterizes ∂_π uniquely. So one has to show existence. We only sketch the construction here, the details are left to the reader. Introduce a formal element ξ with $\xi^2 = \xi\{-1\}$ and $\deg(\xi) = 1$. Define a formal map θ_π by by

$$\theta_\pi(u_1\pi^{i_1}, \dots, u_n\pi^{i_n}) = (i_1\xi + \{\bar{u}_1\}) \cdots (i_n\xi + \{\bar{u}_n\}) \ .$$

We define ∂_π by taking the (right-)coefficient of ξ in θ_π, where we use graded commutativity. This gives a well defined homomorphism

$$\partial_\pi : \mathcal{T}_n^t(A) \longrightarrow K_{n-1}^M(A[t]/(\pi)) \ .$$

So what remains to be shown is that ∂_π factors over the Steinberg relations.
Let $x = (\pi^i u, -\pi^i u)$ be feasible, then

$$\begin{aligned} \theta_\pi(x) &= (i\xi + \{\bar{u}\})(i\xi + \{-\bar{u}\}) \\ &= i\xi\{-1\} - i\xi\{\bar{u}\} + i\xi\{-\bar{u}\} + \{\bar{u}, -\bar{u}\} = 0 \ . \end{aligned}$$

For $i > 0$ and $x = (\pi^i u, 1 - \pi^i u)$ feasible one has

$$\theta_\pi(x) = (i\xi + \{\bar{u}\})\{1\} = 0 \ .$$

For $i < 0$ and $x = (\pi^i u, 1 - \pi^i u)$ feasible one has

$$\begin{aligned} \theta_\pi(x) &= (i\xi + \{\bar{u}\})(i\xi + \{-\bar{u}\}) \\ &= i\xi\{-1\} + i\xi\{-\bar{u}\} - i\xi\{\bar{u}\} + \{\bar{u}, -\bar{u}\} = 0 \ . \end{aligned}$$

Step 3: The filtration $L_d \subset K_n^t(A)$.

Let L_d be the subgroup of $K_n^t(A)$ generated by feasible n-tuples of polynomials of degree at most d. According to step 1 $L_0 = K_n^M(A)$. Moreover from the construction of step 2 we see that if π is of degree d one has $\partial_\pi(L_{d-1}) = 0$.
In order to finish the proof one shows that for $d > 0$

$$L_d/L_{d-1} \longrightarrow \oplus_{\deg(\pi)=d} K_{n-1}^M(A[t]/(\pi)) \tag{14}$$

is an isomorphism.

Step 4: The homomorphism $h_\pi : K_{n-1}^M(A[t]/(\pi)) \to L_d/L_{d-1}$.

For $\deg(\pi) = d$ and $\bar{g} \in A[t]/(\pi)$ let $g \in A[t]$ be the unique representative with $\deg(g) < d$. Then we claim that there exists a unique homomorphism $h_\pi : K_{n-1}^M(A[t]/(\pi)) \to L_d/L_{d-1}$ such that

$$h_\pi(\{\bar{g}_2, \dots, \bar{g}_n\}) = \{\pi, g_2, \dots, g_n\}$$

for $(\pi, g_2, \dots, g_n)$ feasible.

We prove uniqueness first. Let $(\bar{g}_2, \ldots, \bar{g}_n)$ be arbitrary with $\bar{g}_i \in A[t]/(\pi)$ for all i. According to the appendix we can factor each g_i in two polynomials g_i', g_i'' module π if $d > 2$ such that all the 2^{n-1} tuples $(g_2^*, \ldots, g_n^*)$ are feasible ($*$ means primed respectively double primed) and $\deg(g_i^*) = d-1$. In case $d = 2$ we can similarly factor g_i into three polynomials but we leave this to the reader because the argument is essentially the same. So if we have given a homomorphism h_π with the above property we can write

$$h_\pi(\{\bar{g}_2, \ldots, \bar{g}_n\}) = \sum_{*^{n-1}} \{\pi, g_2^*, \ldots, g_n^*\} \ .$$

where the sum is over the 2^{n-1} maps from the set $\{2, \ldots, n\}$ to $\{','' \}$. But the right side obviously does not depend on the the given h_π which shows uniqueness.
Now we show existence. Assume $d > 2$. The case $d = 2$ is similar but one has to factor everything into three polynomials. Given $\{\bar{g}_2, \ldots, \bar{g}_n\} \in K_{n-1}^M(A[t]/(\pi))$ use Theorem 8.1 to choose for every $i = 2, \ldots, n$ a factorization

$$g_i \equiv g_i' g_i'' \mod \pi$$

such that all the tuples $(\pi, g_2^*, \ldots, g_n^*)$ are feasible and $\deg(g_i^*) \le d-1$. Set

$$h_\pi((\bar{g}_2, \ldots, \bar{g}_n)) = \sum_{*^{n-1}} \{\pi, g_2^*, \ldots, g_n^*\} \ .$$

One has to show that this gives a well defined homomorphism

$$(A[t]/(\pi))^{* \otimes n-1} \longrightarrow L_d/L_{d-1} \ .$$

In order to simplify the notation we treat the case $n = 2$. Let

$$g_2 \equiv g' g'' \mod \pi$$

be another generic factorization with $\deg(g^*) = d-1$. By what is proved in the appendix, especially Remark 8.3, we can assume that the tuple $(g_2', g_2'', g', g'', f, \pi)$ is feasible where f is defined by the equation

$$g_2' g_2'' = g' g'' + f\pi$$

The Steinberg relation associated to

$$\frac{g' g''}{g_2' g_2''} + \frac{f\pi}{g_2' g_2''} = 1 \ .$$

shows that

$$\{\pi, g_2'\} + \{\pi, g_2''\} = \{\pi, g_2\} + \{\pi, g_2\} \in L_d/L_{d-1}$$

which implies that we get a well defined map

$$(A[t]/(\pi))^{* \times n-1} \longrightarrow L_d/L_{d-1} \ .$$

Next we will show that this map is in fact multilinear. Again we restrict to $n = 2$. Let $\bar{g}_2 = \bar{p}\,\bar{q} \in (A[t]/(\pi))^*$ and let

$$p \equiv p' p'' \mod \pi \quad \text{ and } \quad q \equiv q' q'' \mod \pi$$

be generic factorizations as above. Then we have

$$\begin{aligned} h_\pi(\bar{p}) + h_\pi(\bar{q}) &= \{\pi, p'\} + \{\pi, p''\} + \{\pi, q'\} + \{\pi, q''\} \\ &= \{\pi, p'q'\} + \{\pi, p''q''\}. \end{aligned}$$

Let f and f' be defined by the equations

$$p'q' + f'\pi = P' \text{ and } p''q'' + f''\pi = P''$$

where $\deg(P'), \deg(P'') < d$. As p', q' or p'', q'' can be chosen generically we can assume that (p', q', f', π, P') and $(p'', q'', f'', \pi, P'')$ are feasible. Similar to the argument given above this implies the equations

$$\{\pi, p'q'\} = \{\pi, P'\} \text{ and } \{\pi, p''q''\} = \{\pi, P''\} \text{ in } L_d/L_{d-1}\ .$$

So we get

$$h_\pi(\bar{p}) + h_\pi(\bar{q}) = \{\pi, P'\} + \{\pi, P''\} = h_\pi(\bar{p}\bar{q})$$

since $\bar{P}'\bar{P}'' = \bar{p}\bar{q}$ is a feasible factorization.
Finally, we show the compatibility with the Steinberg relations. Choosing a factorization $g \equiv g'g''$ mod π such that (π, g', g'') is feasible we get

$$h_\pi(\bar{g} \otimes -\bar{g}) = \{\pi\}(\{g'\} + \{g''\})(\{-g'\} + \{g''\}) = 0\ . \tag{15}$$

In order to show $h_\pi(\bar{g} \otimes (1 - \bar{g})) = 0$ one can assume g generic of degree $d - 1$ in which case it is clear as we can simply lift the Steinberg relation. The fact that we can assume that g is a generic element follows from the five term relation (7,Sublemma 3.6) and (15).

Step 5: $h : \oplus_{\deg(\pi)=d} K^M_{n-1}(A[t]/(\pi)) \to L_d/L_{d-1}$ *is surjective.*

For simplicity we restrict to $d > 2$. The case $d = 2$ can be shown similarly. Consider the symbol $\{p_1, \ldots, p_n\} \in K^t_n(A)$ with $p_i \in A[t]$ prime and $\deg(p_i) \leq d$. Use induction on the number of p_i which are of degree d. We can restrict to $n = 2$. We show that $\{\pi, f\} \in L_d$ lies in the image of this homomorphism for irreducible coprime $\pi, f \in A[t]$ of degree d. As explained in the appendix (modulo the complication that $\deg(f) = d$) choose a generic factorization

$$f = f'\pi + f_1 f_2$$

with $(f'\pi, f, f_1, f_2)$ feasible and $\deg(f_1) = \deg(f_2) = \deg(f') + 1 = d - 1$. The Steinberg relation associated to

$$\frac{f}{f_1 f_2} + \frac{-f'\pi}{f_1 f_2} = 1$$

gives $\{\pi, f\} \in im(h)$ mod L_{d-1}.

Conclusion:

It is obvious that $\partial_\pi \circ h_\pi = 1$. Step 5 shows $\sum_\pi (h_\pi \circ \partial_\pi)$ is the identity on L_d/L_{d-1}. Because for every $d > 0$

$$L_d/L_{d-1} \longrightarrow \oplus_{\deg(\pi)=d} K^M_{n-1}(A[t]/(\pi))$$

is an isomorphism

$$K^t_n(A)/L_0 \longrightarrow \oplus_\pi K^M_{n-1}(A[t]/(\pi))$$

is an isomorphism too. This finishes the proof of Theorem 4.4. □

The relation between the exact sequence (13) and the classical Milnor-Bass-Tate sequence for $F = Q(A)$ is explained in the following proposition.

Proposition 4.7. *The following diagram commutes*

$$\begin{array}{ccccc} K_n^M(A) & \longrightarrow & K_n^t(A,p) & \longrightarrow & \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \\ \downarrow & & \downarrow & & \downarrow \\ K_n^M(F) & \longrightarrow & K_n^M(F(t)) & \longrightarrow & \oplus_\pi K_{n-1}^M(F[t]/(\pi)) \end{array}$$

In the upper row the sum is over all $\pi \in A[t]$ irreducible, monic, and prime to p, in the lower row over all $\pi \in F[t]$ irreducible and monic.

Proof. The commutativity of the left square is clear. For the right square project the lower direct sum onto $K_{n-1}^M(F[t]/(\pi))$. An element

$$\{\pi, g_2, \dots, g_n\} \in K_n^t(A)$$

with $(g_i, \pi) = 1$ maps to $\{\bar{g}_2, \dots, \bar{g}_n\} \in K_{n-1}^M(F[t]/(\pi))$ in any case. □

5. Transfer

In this section we explain how to construct a transfer – also called norm – for finite, étale extensions of semi-local rings with infinite residue fields. Such extensions are exactly those which are of the form $B = A[t]/(\pi)$ with π monic and $\mathrm{Disc}(\pi) \in A^*$.

Definition 5.1. *A polynomial $p \in A[t]$ is called feasible if the highest non-vanishing coefficient of p is invertible in A. It is called irreducible if it cannot be factored nontrivially into polynomials with highest coefficients invertible.*

Because $A[t]$ is not necessarily factorial we generalize Definition 4.1 by attaching as additional data to the p_i, q_i $i = 1, \dots, n$ a factorization up to units into irreducible polynomials with highest coefficients invertible. Furthermore we demand that $\mathrm{Disc}(p_i) \in A^*$ and $\mathrm{Disc}(q_i) \in A^*$. Later we will need the latter conditions to ensure nice functoriality properties for our generalized Milnor K-theory.

Definition 5.2. *An n-tuple of rational functions*

$$(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_n}{q_n}) \in F(t)^n$$

with $p_i, q_i \in A[t]$ feasible together with factorizations of p_i, q_i

$$\begin{aligned} p_i &= a_i p_i^{(1)} \cdots p_i^{(j_i)} \\ q_i &= b_i q_i^{(1)} \cdots q_i^{(j_i)} \end{aligned}$$

with $a_i, b_i \in A^*$ and $p_i^{(j)}, q_i^{(j)}$ monic irreducible is called feasible if:

- *In the obvious sense the p_i/q_i are reduced fractions.*
- *For $i \neq i'$ we have either $p_i^{(j)}, q_i^{(j)}$ equal or coprime to $p_{i'}^{(j')}, q_{i'}^{(j')}$ for $j \in \{1, \dots, j_i\}$ and $j' \in \{1, \dots, j_{i'}\}$.*
- $\mathrm{Disc}(p_i), \mathrm{Disc}(q_i) \in A^*$.

Now that we have the notion of a feasible n-tupel we can immediately generalize Definition 4.2 to get a group $\mathcal{T}_n^{et}(A)$. Furthermore we define the subgroup $St^{et} \subset \mathcal{T}_n^{et}(A)$, in analogy to St, to be generated by feasible n-tupels

$$(p_1, \dots, \frac{p}{q}, \frac{q-p}{q}, \dots, p_n)$$

and

$$(p_1, \dots, \frac{p}{q}, -\frac{p}{q}, \dots, p_n)$$

with $p_i, p, q \in A[t]$ and $(p,q) = 1$, $(q-p,q) = 1$. Here we may attach arbitrary factorizations to $p_i, p, q, q-p$ such that the n-tupel is feasible.

Definition 5.3. *Define*

$$K_n^{et}(A) = \mathcal{T}_n^{et}(A)/St$$

The proof of the next theorem is analogous to the proof of Theorem 4.4.

Theorem 5.4. *There exists a split exact sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^{et}(A) \longrightarrow \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \longrightarrow 0$$

where the direct sum is over all monic irreducible $\pi \in A[t]$ with $\mathrm{Disc}(\pi) \in A^*$.

Now one defines a transfer as in the field case

$$N_{B/A} : K_n^M(B) \longrightarrow K_n^M(A)$$

by setting:

Definition 5.5. *For $x \in K_n^M(B)$ choose $x' \in K_{n+1}^{et}(A)$ with $\partial_\pi(x') = x$ and $\partial_{\pi'}(x') = 0$ for all monic irreducible $\pi' \neq \pi \in A[t]$. Define*

$$N_{B/A}(x) = -\partial_\infty(x')$$

where $\partial_\infty : K_{n+1}^{et}(A) \to K_n^M(A)$ is the infinite residue symbol defined analogously to the infinite residue symbol in the proof of Theorem 4.4 Step 1.

Assume given an arbitrary homomorphism – not necessarily local – of semi-local rings $i : A \to A'$. Fix as additional data a factorization into monic irreducible polynomials for every polynomial $i(p) \in A'[t]$ where $p \in A[t]$ is monic irreducible. Let $\pi \in A[t]$ be a monic irreducible polynomial with $\mathrm{Disc}(\pi) \in A^*$ and let $i(\pi) = \prod_j \pi_j$ be the associated complete factorization. Denote $B = A[t]/(\pi)$ and $B'_j = A'[t]/(\pi_j)$.
Proposition 4.7 generalizes to:

Proposition 5.6. *The following diagram of exact sequences from Theorem 5.4 commutes:*

$$\begin{array}{ccccc} K_n^M(A) & \longrightarrow & K_n^{et}(A) & \longrightarrow & \oplus_\pi K_{n-1}^M(A[t]/(\pi)) \\ \downarrow & & \downarrow & & \downarrow \\ K_n^M(A') & \longrightarrow & K_n^{et}(A') & \longrightarrow & \oplus_{\pi'} K_{n-1}^M(A'[t]/(\pi')) \end{array}$$

The right vertical map is defined by the natural homomorphism

$$K_{n-1}^M(A[t]/(\pi)) \longrightarrow \oplus_{\pi_j} K_{n-1}^M(A'[t]/(\pi_j)) \ .$$

One should remark that the existence of the middle vertical arrow is guaranteed by the condition that all our polynomials have non-vanishing discriminant.
Now our main compatibility result states:

Proposition 5.7. *The diagram*

$$\begin{array}{ccc} K_n^M(B) & \longrightarrow & \oplus_j K_n^M(B'_j) \\ {\scriptstyle N_{B/A}}\big\downarrow & & \big\downarrow{\scriptstyle \oplus_j N_{A'_j/A'}} \\ K_n^M(A) & \longrightarrow & K_n^M(A') \end{array}$$

is commutative.

Remark 5.8. *By Remark 8.4 and the above construction for every $n \geq 0$ there clearly exists an $M_n \in \mathbb{N}$ such that for $B = A[t]/(\pi)$, $\deg(\pi) = 2, 3$, π irreducible and monic, $\mathrm{Disc}(\pi) \in A^*$ and A a semi-local ring with more than M_n elements in each residue field there exist norms*

$$N : K_n^M(B) \to K_n^M(A)$$

that satisfy $N \circ i_* = \deg(\pi)$ where $i : A \to B$ is the embedding.

In principle one could go through the construction of the transer in order to determine a possible choice for M_n. In this paper we will not be concerned with this problem.

6. Main theorem

The main result is:

Theorem 6.1. *Let A be a regular connected semi-local ring containing a field with quotient field F. Assume that each residue field of A has more than M_n elements (see Remark 5.8). Then the map*

$$i_n : K_n^M(A) \longrightarrow K_n^M(F)$$

is universally injective.

For the convenience of the reader we recall the definition of universal injectivity from [4].

Definition 6.2. *Let*

$$A' \longrightarrow A \longrightarrow A''$$

be a sequence of abelian groups. We say this sequence is universally exact if

$$F(A') \longrightarrow F(A) \longrightarrow F(A'')$$

is exact for every additive functor $F : \mathbf{Ab} \to \mathbf{B}$ which commutes with filtering small colimits. Here we assume $\mathbf{B}$ is an abelian category satisfying AB5 (see [9]*).*

For the non-smooth case of the main theorem we need Néron-Popescu desingularization [28]:

Lemma 6.3 (Popescu)**.** *Let $h : A \to B$ be a regular homomorphism of noetherian rings, that is the geometric fibers of h are regular. Then h is the filtering direct limit*

$$h = \varinjlim h_i$$

of smooth morphisms $h_i : A_i \to B_i$ of noetherian rings A_i, B_i.

Proof. First we prove ordinary injectivity under the assumption that k is an infinite perfect field and A is the semi-local ring associated to a finite set of closed points of a smooth, affine variety X/k of dimension d. In order to prove this special case use induction on d. The case $d = 0$ is trivial.
Suppose given $x \in K_n^M(A)$ such that $i_n(x) = 0$. Then there is $0 \neq f \in A$ such that $i'_n(x) = 0$ with $i'_n : K_n^M(A) \longrightarrow K_n^M(A_f)$ the canonical map.
Use Gabber's geometric presentation theorem [4] to construct a k-morphism $\phi : X \to \mathbb{A}_k^d$. Let A' be the semi-local ring at the images under ϕ of the points correspondig to A. Denote these points by $y_1, \ldots, y_l \in \mathbb{A}_k^d$. After shrinking X we can assume ϕ satisfies the following properties:

(i) The map $V(f) \to \mathbb{A}_k^d$ is an embedding.
(ii) ϕ is étale.
(iii) If $f' \in A'$ is chosen according to (i) such that $A/(f) \cong A'/(f')$, then $A/(f) = A \otimes_{A'} A'/(f')$.

Consider the commutative diagram

$$\begin{array}{ccc} K_n^M(A') & \longrightarrow & K_n^M(A'_{f'}) \\ \big\downarrow & & \big\downarrow \\ K_n^M(A) & \longrightarrow & K_n^M(A_f) \end{array}$$

Theorem 3.1 shows that it is co-Cartesian. According to a well known property of co-Cartesian squares the lower horizontal arrow is injective if the upper horizontal arrow is injective.
So we have to prove

$$i_n : K_n^M(A') \longrightarrow K_n^M(k(t_1, \ldots, t_d))$$

is injective.
Let again x be in the kernel of this homomorphism and denote by $p_1, \ldots, p_m \in k[t_1, \ldots, t_d]$ the irreducible different polynomials appearing in the symbols of x, $p_i \in A'^*$.
Denote for $i = 1, \ldots, m$ by $W_i \subset V(p_i)$ the join of the singular locus of $V(p_i)$ with $\bigcup_{j \neq i} V(p_i) \cap V(p_j)$. Because we assumed k to be perfect $\dim(W_i) < d - 1$.
Use a slight generalization of Noether normalization to choose a linear projection

$$p : \mathbb{A}_k^d \longrightarrow \mathbb{A}_k^{d-1}$$

such that $p|_{V(p_j)}$ is finite and $p(y_i) \notin p(W_j)$ for $i = 1, \ldots, l$ and $j = 1, \ldots, m$.
Let A'' be the semi-local ring associated to the points $p(y_1), \ldots, p(y_l) \in \mathbb{A}_k^{d-1}$. Then $A'' \subset A'$ is a local ring extension and because $V(p_i)$ is finite integral over A'' one sees that $p_i \in A''[t]$ can be chosen to be monic and irreducible. Choose a monic $q \in A''[t]$ such that $V(q) \cap p^{-1}(p(y_i))$ consists exactly of the points from $\{y_1, \ldots, y_l\}$ which are in the fibre over $p(y_i)$ for all $i = 1, \ldots, l$. It follows that $(q, p_i) = 1$ for $i = 1, \ldots, m$.
There exists a natural map $K_n^t(A'', q) \to K_n^M(A')$. Now x is induced by an element $x' \in K_n^t(A'', q)$. Consider the commutative diagram with exact rows from Proposition

4.7

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_n^M(A'') & \longrightarrow & K_n^t(A'',q) & \xrightarrow{\alpha} & \oplus_\pi K_{n-1}^M(A''[t]/(\pi)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & K_n^M(F) & \longrightarrow & K_n^M(F(t)) & \longrightarrow & \oplus_\pi K_{n-1}^M(F[t]/(\pi)) & \longrightarrow & 0
\end{array}$$

where the notation $F = Q(A'')$ is used. By assumption $\beta(x') = 0$. By induction the relevant summands of δ are injective so that $\alpha(x') = 0$. But then x' comes from an element $x'' \in K_n^M(A'')$ with $\gamma(x'') = 0$. It follows again by induction that $x'' = 0$ and $x' = 0$.
This finishes the proof of Theorem 5.1, without the universality property, in case A is the semi-local ring at closed points of an smooth, affine variety X/k and k is infinite and perfect.

Next we show injectivity for a semi-local ring A corresponding to an arbitrary system of points $y_1, \dots, y_l$ of a smooth affine variety X over an infinite perfect field k. Let F be the quotient field of A. Given

$$x \in \ker(K_n^M(A) \longrightarrow K_n^M(F))$$

choose for every y_i, $i = 1, \dots, l$ a closed point $y_i' \in \overline{\{y_i\}}$ such that if A' denotes the semi-local ring corresponding to these points x is induced by an $x' \in K_n^M(A')$ under the natural map $K_n^M(A') \to K_n^M(A)$.
Because

$$x' \in \ker(K_n^M(A') \longrightarrow K_n^M(F))$$

and this map is injective we deduce $x' = 0$ and $x = 0$.

At this point we can write down the isomorphism

$$K_n^M(A) = H^n(Spec(A), \mathbb{Z}(n))$$

for a ring A as in the last paragraph, as will be explained in Theorem 7.6. But as

$$H^n(Spec(A), \mathbb{Z}(n)) \longrightarrow H^n(Spec(F), \mathbb{Z}(n))$$

is universally injective according to [4] the corresponding injection of Milnor K-groups is universally injective too.

For the general case of our theorem we use Néron-Popescu desingularization, Lemma 6.3. In fact one has to show

$$K_n^M(A) \longrightarrow K_n^M(A_f)$$

is universally injective for $0 \neq f \in A$. As A is the filtered inductive limit of smooth semi-local rings of geometric type over a prime field, it is sufficient to restrict to the case in which A is the semi-local ring at some points of a smooth, affine variety X over a prime field k_0 and the residue fields have more than M_n elements. For the argument below take M_n from Remark 5.8.

If $\mathrm{char}(k_0) > 0$ one has to use a norm trick to reduce to the case of a ground field which is an infinite algebraic extension of k_0. Let $k_1 \subset A$ be the algebraic closure of k_0 in A. Now argue as follows:
Fix $p = 2$ or $p = 3$. Choose a tower of finite extensions $k_1 \subset k_2 \subset k_3 \subset \cdots \subset k_\infty$ with $k_\infty = \cup_i k_i$ and $\dim_{k_i}(k_{i+1}) = p$, $i = 1, 2, \ldots$
From Remark 5.8 one deduces the existence of norms

$$N : K_n^M(A \otimes_{k_1} k_{i+1}) \longrightarrow K_n^M(A \otimes_{k_1} k_i)$$

which satisfy $N \circ i_* = p$ for the natural map $i : A \otimes k_i \to A \otimes k_{i+1}$.
Consider the commutative diagram

$$\begin{array}{ccc} K_n^M(A \otimes_{k_1} k_\infty) & \longrightarrow & K_n^M(F \otimes_{k_1} k_\infty) \\ \alpha\Big\uparrow & & \Big\uparrow \\ K_n^M(A) & \xrightarrow{\beta} & K_n^M(F) \end{array}$$

with $F = Q(A)$. The upper arrow is universally injective according to what we proved above. Because of the existence of a norm $\alpha \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ is universally injective so that $\beta \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ is universally injective.
This implies β is universally injective, since $p = 2$ or 3 and for a functor F as in Definition 6.2 and an abelian group G we have

$$F(G \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]) = F(G) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \ .$$

$\square$

7. Applications

We will give some consequences of Theorem 6.1.

Assumption: All schemes and rings in this section up to Theorem 7.6 are excellent.

Recall that Kato constructed a Gersten complex of Zariski sheaves for Milnor K-theory of a scheme X

$$0 \to \mathcal{K}_n^M|_X \to \oplus_{x \in X^{(0)}} i_{x\,*}(K_n^M(x)) \to \oplus_{x \in X^{(1)}} i_{x\,*}(K_{n-1}^M(x)) \to \cdots \tag{16}$$

where $K_*^M(x) := K_*^M(\mathbf{k}(x))$ and i_x is the embedding of the point x [14].
Milnor K_n^M of a field coincides with (n, n)-motivic cohomology [26], [29] – for the latter the exactness of the Gersten complex is well known [4] if X is smooth over a perfect field. Moreover the differentials of (16) are equal to the ones constructed from the coniveau spectral sequence in motivic cohomology modulo a sign. This implies that (16) is exact except at the first two places if X is regular and of algebraic type over an arbitrary field. An elementary proof of this fact can be found in [24].
The question whether (16) is exact at the second place was settled independently by Gabber [7] and Elbaz-Vincent/Müller-Stach [6], for a short proof see [16].
From Theorem 6.1 and Panin's method [22] we conclude the Gersten conjecture is true in an equicharacteristic context:

Theorem 7.1 (Gersten conjecture)**.** *The Gersten complex (16) for Milnor K-theory is exact if X is regular, contains a field, and all residue fields of X contain more than M_n elements (see Remark 5.8).*

For the definition of universal exactness see Definition 6.2. Our proof follows [22] closely.

Proof. After Theorem 6.1 and our remarks above it is sufficient to prove the exactness in degrees > 0 of the complex of Zariski sheaves

$$g_n(X) = (\oplus_{x\in X^{(0)}} i_{x\,*}(K_n^M(x)) \longrightarrow \oplus_{x\in X^{(1)}} i_{x\,*}(K_{n-1}^M(x)) \longrightarrow \cdots)$$

if $X = Spec(A)$ with A regular and equicharacteristic. Here $\oplus_{x\in X^{(0)}} i_{x\,*}(K_n^M(x))$ is understood to be placed in degree zero. We use induction on $d = \dim(X)$.
Let $f \in A$ be a local parameter and $Z = Spec(A/(f))$. Then we have a short exact sequence

$$0 \longrightarrow g_{n-1}(Z)[-1] \longrightarrow g_n(X) \longrightarrow g_n(X_f) \longrightarrow 0 \tag{17}$$

as in [22]. Our induction assumption implies that $H^i(g_{n-1}[-1](Z)) = 0$ for $i \geq 2$ and $H^1(g_{n-1}[-1](Z)) = K_{n-1}^M(Z)$. Furthermore, because $\dim(X_f) < d$, $g_n(X_f)$ is the global section complex associated to a flabby resolution of $\mathcal{K}_n^M$. In other words:

$$H^i(g_n(X_f)) = H^i(X_f, \mathcal{K}_n^M)\ .$$

The latter cohomology groups can be calculated by going down to a smooth world:

Lemma 7.2. *We have $H^i(X_f, \mathcal{K}_n^M) = 0$ for $i > 0$ and $H^0(X_f, \mathcal{K}_n^M) = K_n^M(A_f)$.*

Proof. By the following continuity result and Néron-Popescu desingularization, Lemma 6.3, we can assume X to be essentially smooth over a prime field with a residue field with more than M_n elements.

Sublemma 7.3. *Let X_i be a filtering inverse limit of affine noetherian schemes with $\lim_{\rightarrow} X_i = X$ noetherian and let $(F_i)_i$ be a compatible system of Zariski sheaves on the schemes X_i with limit sheaf F on X. Then the natural map*

$$\lim_{\rightarrow} H^n(X_i, F_i) \longrightarrow H^n(X, F)$$

is an isomorphism.

But then, reading our argument backwards and using the fact that we know from [24], [7], and Theorem 6.1 that the Gersten conjecture is true for smooth varieties, we have $H^i(g_n(X_f)) = 0$ for $i > 0$ because of the long exact cohomology sequence associated to (17). Furthermore (17) induces a short exact sequence

$$0 \longrightarrow K_n^M(A) \longrightarrow \mathcal{K}_n^M(X_f) \longrightarrow K_{n-1}^M(A/(f)) \longrightarrow 0\ .$$

As a consequence of Theorem 6.1 we get an analogous sequence with $\mathcal{K}_n^M(A_f)$ replaced by $K_n^M(A_f)$:

Sublemma 7.4. *The canonical sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^M(A_f) \longrightarrow K_{n-1}^M(A/(f)) \longrightarrow 0$$

is exact for an arbitrary equicharacteristic regular local ring A and irreducible element f.

Proof. The injectivity of $K_n^M(A) \longrightarrow K_n^M(A_f)$ follows from Theorem 6.1. The rest is elementary and left to the reader. □

Putting the last two short exact sequences together we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n^M(A) & \longrightarrow & K_n^M(A_f) & \longrightarrow & K_{n-1}^M(A/(f)) & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & K_n^M(A) & \longrightarrow & \mathcal{K}_n^M(X_f) & \longrightarrow & K_{n-1}^M(A/(f)) & \longrightarrow & 0\,. \end{array}$$

Finally, the five-lemma shows

$$H^0(X_f, \mathcal{K}_n^M) = \mathcal{K}_n^M(X_f) = K_n^M(A_f)\,.$$

□

The long exact cohomology sequence associated with (17) gives, inserting the calculations of Lemma 7.2, $H^i(g_n(X)) = 0$ for $i > 1$ and the exact sequence

$$0 \longrightarrow H^0(g_n(X)) \longrightarrow K_n^M(A_f) \xrightarrow{\partial} K_{n-1}^M(A/(f)) \longrightarrow H^1(g_n(X)) \longrightarrow 0\,.$$

As, according to Sublemma 7.4, ∂ is surjective and has kernel $K_n^M(A)$ this finishes the proof of Theorem 7.1. □

Kato's original motivation for studying the Gersten complex was to obtain an elementary generalization of the formula

$$H^1(X, \mathcal{O}_X^*) = CH^1(X)$$

by means of Milnor K-theory. He proved the following fact in case $n = \dim(X)$ and X is smooth of finite type over a Dedekind ring [14]. Except for Kato's result the theorem was previously known up to torsion [25] and for $n = 1, 2$ due to Quillen [23] as Quillen K-theory coincides with Milnor K-theory for local rings with infinite residue fields in degree 2 by van der Kallen's theorem, compare for example [20].

Theorem 7.5 (Bloch formula)**.** *There is a canonical isomorphism*

$$H^n(X, \mathcal{K}_n^M) \cong CH^n(X)\,.$$

for every $n \geq 0$ if X is as in Theorem 7.1.

Furthermore from the exactness of the Gersten complex one deduces one of the remaining Beilinson conjectures on motivic cohomology [18], [3].

Theorem 7.6 (Beilinson's conjecture)**.** *For Voevodsky's motivic complexes of Zariski sheaves $\mathbb{Z}(n)$ [26] on the category of smooth schemes over an infinite field there is an isomorphism*

$$\mathcal{K}_n^M \xrightarrow{\sim} \mathcal{H}^n(\mathbb{Z}(n)) \tag{18}$$

for all $n \geq 0$.

Proof. Voevodsky constructed such motivic complexes $\mathbb{Z}(n)$ whose n-th cohomology sheaf over a field is isomorphic to Milnor K-theory [26]. This and the morphism (up to sign) of exact Gersten complexes of sheaves from Milnor K-theory to motivic cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_n^M|_X & \longrightarrow & \oplus_{x\in X^{(0)}} i_{x*}(K_n^M(x)) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H}^n(\mathbb{Z}(n))|_X & \longrightarrow & \oplus_{x\in X^{(0)}} i_{x*}(H^n(x,\mathbb{Z}(n))) & \longrightarrow & \cdots \end{array}$$

proves the theorem. □

Conjecture 7.7 (Bloch-Kato). *The norm residue homomorphism*

$$\chi_n : K_n^M(E)/l \longrightarrow H^n_{et}(E, \mu_l^{\otimes n})$$

is an isomorphism for all fields E *whose characteristic does not divide* l *and* $n \geq 0$.

A proof of the Bloch-Kato conjecture has been announced by Voevodsky and Rost [33]. Marc Levine [17] and Bruno Kahn [12] conjectured the following generalized version of the Bloch-Kato conjecture. Levine showed even before the advent of modern motivic cohomology that it implies a form of the Quillen-Lichtenbaum conjecture.

Theorem 7.8 (Levine's Bloch-Kato conjecture). *Assume the Bloch-Kato conjecture. The norm residue homomorphism*

$$\chi_n : K_n^M(A)/l \longrightarrow H^n_{et}(A, \mu_l^{\otimes n})$$

is an isomorphism for $n > 0$ *and all semi-local rings* A *containing a field* k *of characteristic not dividing* l *with* $|k| > M_n$.

Proof. Assume first that A is a smooth semi-local ring of geometric type over k. In this case the theorem follows from the morphsim (up to a sign) of universally exact Gersten complexes, $X = Spec(A)$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_n^M(A)/l & \longrightarrow & K_n^M(Q(A))/l & \longrightarrow & \oplus_{x\in X^{(1)}} K_{n-1}^M(x)/l \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^n_{et}(A,\mu_l^{\otimes n}) & \longrightarrow & H^n_{et}(Q(A),\mu_l^{\otimes n}) & \longrightarrow & \oplus_{x\in X^{(1)}} H^n_{et}(\mathbf{k}(x),\mu_l^{\otimes n-1}) \end{array}$$

The general cases uses a trick coined by Hoobler [10]. First of all because both Milnor K-theory and étale cohomology are locally of finite presentation we can assume A to be of geometric type over k. Let $B \to A$ be surjective local morphism of semi-local rings with kernel I such that (B, I) is a henselian pair and B is ind-smooth over k. The homomorphism

$$K_n^M(B)/l \longrightarrow K_n^M(A)/l$$

is surjective. In [8] Gabber proves:

Lemma 7.9 (Gabber).

$$H^n_{et}(B, \mu_l^{\otimes n}) \longrightarrow H^n_{et}(A, \mu_l^{\otimes n})$$

is an isomorphism.

Now the problem is reduced to the smooth case by the following commutative diagram

$$\begin{array}{ccc} K_n^M(B)/l & \longrightarrow & K_n^M(A)/l \\ \downarrow & & \downarrow \\ H^n_{et}(B,\mu_l^{\otimes n}) & \longrightarrow & H^n_{et}(A,\mu_l^{\otimes n}) \end{array}$$

□

For a local ring A let $W(A)$ be the Witt ring and I_A the fundamental ideal.

Theorem 7.10 (Generalized Milnor conjecture)**.** *Assume A is a local ring and contains a field k of characteristic different from two with $|k| > M_n$. Then the natural map*

$$K_n^M(A)/2 \longrightarrow I_A^n/I_A^{n+1}$$

is an isomorphism for $n \geq 0$.

Proof. Assume first that A is a smooth semi-local ring of geometric type over k, $X = Spec(A)$. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_n^M(A)/2 & \longrightarrow & K_n^M(Q(A))/2 & \longrightarrow & \oplus_{x\in X^{(1)}} K_{n-1}^M(x)/2 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_A^n/I_A^{n+1} & \longrightarrow & I^n_{Q(A)}/I^{n+1}_{Q(A)} & \longrightarrow & \oplus_{x\in X^{(1)}} I_x^{n-1}/I_x^n \end{array}$$

where the exactness of the lower sequence follows from the exactness of the upper sequence. This is because the left vertical map is surjective for elementary reasons and the other vertical maps are isomorphisms by Voevodsky's theorem [21]. The exactness of the upper sequence is nothing but Theorem 7.1.
A diagram chase proves the theorem if A is essentially smooth over k. Choosing $B \to A$ as in the proof of the last theorem we have:

Lemma 7.11. *The natural homomorphism*

$$I_B^n/I_B^{n+1} \to I_A^n/I_A^{n+1}$$

is an isomorphism.

Proof. One can show that $W(B) \to W(A)$ is an isomorphism [1, Chapter V (1.5)]. The lemma follows immediately. □

The following commutative diagram finishes the proof by reducing to the smooth case

$$\begin{array}{ccc} K_n^M(B)/2 & \longrightarrow & K_n^M(A)/2 \\ \downarrow & & \downarrow \\ I_B^n/I_B^{n+1} & \longrightarrow & I_A^n/I_A^{n+1} \end{array}$$

□

Finally, it follows that the transfer for Milnor K-groups of étale finite extensions B/A of semi-local rings constructed in Section 5 does not depend on any choice made if the rings are equicharacteristic.

Theorem 7.12. *If A contains an infinite field the transfer*

$$N_{B/A} : K_n^M(B) \longrightarrow K_n^M(A)$$

does not depend on the chosen generator of B over A and is functorial.

Proof. Let $B = A[t]/(\pi)$. Choose a regular, semi-local ring A' containing an infinite field and a map $i : A' \to A$ such that there exists a polynomial $\pi' \in A'[t]$ with $i(\pi') = \pi$. According to Proposition 5.3 and by choosing A' large we have to show that

$$N_{B/A} : K_n^M(B') \longrightarrow K_n^M(A')$$

does not depend on the generator of $B' = A'[t]/(\pi')$ over A'. The theorem is reduced to the case in which A is a field by the diagram

$$\begin{array}{ccc} K_n^M(B') & \longrightarrow & K_n^M(Q(B')) \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ K_n^M(A') & \longrightarrow & K_n^M(Q(A')) \end{array}$$

which according to Proposition 5.3 is commutative and by Theorem 6.1. Finally, one uses that Kato proved the theorem in the field case [13]. □

8. Appendix

In this appendix we generalize a factorization result used by Gabber [7] in his proof of the surjectivity of the homomorphism of sheaves

$$\mathcal{K}_n^M \longrightarrow \mathcal{H}^n(\mathbb{Z}(n))$$

on the big Zariski site of smooth varieties over an infinite field.
We use the expression feasible polynomial to mean a polynomial with highest non-vanishing coefficient invertible.

Theorem 8.1. *Let A be a semi-local ring with infinite residue fields, $\pi \in A[t]$ of degree $d > 2$ a monic polynomial. Then every $[p] \in (A[t]/(\pi))^*$, $p \in A[t]$ and $\deg(p) < d$, can be written as*

$$p = f\,\pi + p_1\,p_2$$

with $f, p_1, p_2 \in A[t]$ feasible, $\deg(f) = d-2$, $\deg(p_i) = d-1$.
Furthermore we can achieve that in (p, f, π, p_1, p_2) every two elements are coprime and

$$\begin{aligned} \mathrm{Disc}_{2d-2}(p_1\,p_2) &\in A^* \\ \mathrm{Disc}_{d-2}(f) &\in A^* \,. \end{aligned}$$

Here for a polynomial $f \in A[t]$ of degree d we denote by $\mathrm{Disc}_d(f)$ its discriminant and for two polynomials p_1, p_2 of degrees d_1, d_2 we denote by $\mathrm{Res}_{d_1,d_2}(p_1, p_2)$ their resultant.

Proof. Let m be the Jacobson radical of A and denote the image of a polynomial $g \in A[t]$ in $(A/m)[t]$ by $\bar{g}$. We first reduce to the case that A is a field. Suppose the result is known in this case and

$$\bar{p} = \bar{f}\,\bar{\pi} + \bar{p_1}\,\bar{p_2}$$

is such a factorization in A/m. We can choose $p_1 \in A[t]$ such that $\deg(p_1) = d-1$ and such that $\bar{p}_1$ satisfies the conditions of the theorem. Then $\mathrm{Res}_{d,d-1}(\pi, p_1) \in A^*$ and we see that we can choose $p_2, f \in A[t]$, $\deg(p_2) = d-1$, $\deg(f) = d-2$, such that

$$p = f\,\pi + p_1\,p_2\ .$$

The conditions of the theorem on $f\,\pi, p_1, p_2$ are now automatically satisfied.
Now we suppose A is an infinite field. We identify the space of polynomials of degree at most $d-1$ with $\mathbb{A}_A^{d-1}$. Then the set of such polynomials prime to an arbitrary non-vanishing polynomial in $A[t]$ is dense and open in $\mathbb{A}_A^{d-1}$. Furthermore it is clear that every dense open set in $\mathbb{A}_A^{d-1}$ contains an A-rational point, moreover the intersection of finitely many dense open sets is dense and open.
Therefore it is immediately clear that we always have a factorization

$$p = f\,\pi + p_1\,p_2\ .$$

where p_1, p_2 are of degree $d-1$ and $\mathrm{Disc}_{d-1}(p_1), \mathrm{Disc}_{d-1}(p_2) \neq 0$ (f is then automatically of degree $d-2$). Next we show that we can choose such a factorization generically so as to satisfy $\mathrm{Disc}_{2d-2}(p_1\,p_2) \neq 0$.
Case $\mathrm{char}(A) = 0$*:* The idea is to give a factorization with $p_1(0), p_2(0) \neq 0$ such that

$$\mathrm{Res}_{d-1,d-1}(t^{d-1}p_1(1/t), t^{d-1}p_2(1/t)) \neq 0 \tag{19}$$

Choose $x_0 \in A$ such that $p(x_0), \pi(x_0) \neq 0$ and let $f = p(x_0)/\pi(x_0)$. We can further assume $p(0) - f\,\pi(0) \neq 0$ and $p'(x_0) - f\,\pi'(x_0) \neq 0$ (the latter because $\mathrm{char}(A) = 0$).
Now let $p_1 = t - x_0$. It is obvious that (19) is satisfied for this choice and therefore it is generically satisfied. But generically p_1, p_2 are of degree $d-1$. In this case (19) is equivalent to

$$\mathrm{Res}_{d-1,d-1}(p_1, p_2) \neq 0\ .$$

This shows we generically have $\mathrm{Disc}_{2d-2}(p_1\,p_2) \neq 0$ and proves the theorem in case $\mathrm{char}(A) = 0$.
Case $\mathrm{char}(A) \neq 0$*:* The above proof works except in case $p' = \pi' = 0$. Now one can use a similar argument in order to show (19) is satisfied generically. We take f to be of degree 1 and two different points $x_0, x_1 \in A$ such that $p(x_0) - f(x_0)\,\pi(x_0) = 0$, $p(x_1) - f(x_1)\,\pi(x_1) = 0$, $\pi(x_0) \neq 0$, $\pi(x_1) \neq 0$, $p(0) - f(0)\,\pi(0) \neq 0$. Let $p_1 = (t - x_0)(t - x_1)$.
The fact that $f \in A[t]$ can be assumed to satisfy $\mathrm{Disc}_{d-2}(f) \in A^*$ follows because over the algebraic closure of A there clearly exists at least one such factorization with no other conditions imposed so that $\mathrm{Disc}_{d-2}(f) \in A^*$ is generically satisfied.

□

There is an equivalent theorem for $\deg(\pi) = 2$ which we state below. Its proof is completely analogous.

Theorem 8.2. *Let A be a semi-local ring with infinite residue fields, $\pi \in A[t]$ a monic polynomial of degree two, $p \in A[t]$ an element coprime to π with $\deg(p) < 2$. Then there exists a factorization*

$$p = f\,\pi + p_1\,p_2\,p_3$$

with f, p_1, p_2, p_3 feasible,

$$\deg(f) = 1, \deg(p_1) = 1, \deg(p_2) = 1, \deg(p_3) = 1$$

$\mathrm{Disc}_3(p_1\, p_2\, p_3) \in A^*$ *and such that in $(p, f, \pi, p_1, p_2, p_3)$ each two elements are coprime.*

Remark 8.3. *One can in fact show that given a monic polynomial $g \in A[t]$ the factorization in Theorem 8.1 and 8.2 can be chosen such that $f, p_1, p_2(, p_3)$ are coprime to g or even such that their residue classes modulo the maximal ideal of A are generic in their moduli spaces as explained in the proof of Theorem 8.1.*

Remark 8.4. *For given $d_1 > 2, d_2 > 0$ there exists an integer M such that a factorization (coprime in the above sense to a monic $g \in A[t]$ of degree d_2) as in Theorem 8.1 for any monic π of degree d_1 and any $p \in A[t]$ of degree smaller d_1 exists if the number of elements in each residue field of A is greater than M.*
Similarly for Theorem 8.2.

Acknowledgment

Many of the ideas of this paper were developed during the work on my diploma thesis at the University of Mainz. I would like to thank Stefan Müller-Stach, the advisor for my diploma thesis, who contributed many ideas exploited here.
Also I am deeply indebted to Burt Totaro for supporting me with a lot of mathematical improvements and the opportunity to work in Cambridge where this paper was written. Finally, I would like to thank Uwe Jannsen for his comments which helped to improve the paper considerably.

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