Cohomology of local systems

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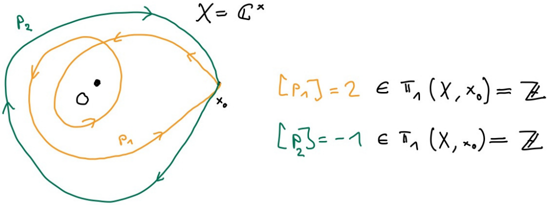
Fundamental group

X 'nice' topological space (e.g. finite, connected CW-complex), $x_0 \in X$.

$\pi_1(X, x_0)$ fundamental group consists of continuous maps $p\colon [0,1] \to X$ with $p(0) = p(1) = x_0$ up to homotopy (=deformation).



Example

$X = \mathbb{C}^\times \simeq S^1 \rightsquigarrow \pi_1(X, x_0) = \mathbb{Z}$
$X = \mathbb{C} \setminus \{0, 1\} \rightsquigarrow \pi_1(X, x_0) =$ free group generated by the loops around 0 and 1.

Representations of groups

A representation of rank r of a group G is a group homomorphism $\rho\colon G \to \mathrm{GL}_r(\mathbb{C})$.

Representations $\rho_1, \rho_2\colon G \to \mathrm{GL}_r(\mathbb{C})$ isomorphic if there is $B \in \mathrm{GL}_r(\mathbb{C})$ such that $\rho_1(g) = B\rho_2(g)B^{-1}$ $(g \in G)$.

$\rho\colon G \to \mathrm{GL}_r(\mathbb{C})$ is semi-simple if for ρ-stable $V \subset \mathbb{C}^r$ there is ρ-stable complement W, $\mathbb{C}^r = V \oplus W$.

Example

For $G = \mathbb{Z}^2$

$$\{\text{reps. } G \to \mathrm{GL}_r(\mathbb{C})\} \cong \{(B_1, B_2) \in \mathrm{GL}_r(\mathbb{C})^2 \mid B_1 \cdot B_2 = B_2 \cdot B_1\}$$
$$\rho \mapsto (\rho(1,0), \rho(0,1)).$$

ρ semi-simple $\Leftrightarrow$ B_1, B_2 diagonalizable.

Local Systems

Local system $\mathcal{L}$ on X = covering $\mathcal{L} \to X$ such that fibre $\mathcal{L}_x$ is $\mathbb{C}$-vector space of dimension r ($x \in X$).

Example

The solutions of the hypergeometric differential equation

$$z(1-z)\frac{d^2f}{dz^2} + (c-(a+b+1)z)\frac{df}{dz} - abf = 0$$

form a local systems of rank 2 on $\mathbb{C} \setminus \{0,1\}$.

Fundamental theorem of covering spaces

There is a natural identification

$$\{\text{local systems } \mathcal{L}/X\}/\text{iso.} \cong \mathrm{Hom}(\pi_1(X,x_0), \mathrm{GL}_r(\mathbb{C}))/\mathrm{GL}_r(\mathbb{C}).$$

Algebraic varieties

$P_1, \ldots, P_s \colon \mathbb{C}^N \to \mathbb{C}$ polynomials
$\rightsquigarrow$ an affine algebraic variety $X = \{x \in \mathbb{C}^N \mid P_i(x) = 0\}$.

Restricting polynomials $\mathbb{C}^N \to \mathbb{C}$ to X gives algebraic functions $\mathcal{O}(X) \subset \{X \to \mathbb{C}\}$.

Example

$\mathbb{C} \setminus \{0, 1\}$ can be identified with the affine algebraic variety

$$\{(x, y, z) \in \mathbb{C}^3 \mid xy = 1, (x-1)z = 1\}$$

by $x \mapsto (x, 1/x, (1/(x-1))$.

Topology of algebraic varieties

An algebraic variety $X \subset \mathbb{C}^N$ with the subspace topology is 'nice' and the fundamental group $\pi_1(X, x_0)$ is finitely presented $\langle g_1, \ldots, g_n | r_j \rangle$.

The character variety

Finitely presented group $\pi = \langle g_1, \dots, g_n | r_j \rangle \rightsquigarrow$

$$\mathrm{Hom}(\pi, \mathrm{GL}_r(\mathbb{C})) = \{\underline{B} \in \mathrm{GL}_r(\mathbb{C})^n \mid r_j(\underline{B}) = 0\}$$

affine algebraic variety $\mathrm{Ch}^{\pi,\square}_{r,\mathbb{C}}$ parameterizing all framed representations of π.

$\mathcal{O}(\mathrm{Ch}^{\pi,\square}_{r,\mathbb{C}})^{\mathrm{GL}_r(\mathbb{C})}$ algebraic functions invariant by conjugation $\rightsquigarrow$ character variety

$$\mathrm{Ch}^{\pi}_{r,\mathbb{C}} = \mathrm{Ch}^{\pi,\square}_{r,\mathbb{C}} // \mathrm{GL}_r(\mathbb{C}) = \mathrm{Spm}\, \mathcal{O}(\mathrm{Ch}^{\pi,\square}_{r,\mathbb{C}})^{\mathrm{GL}_r(\mathbb{C})}$$

Invariant theory

(1) The map $\mathrm{Hom}(\pi, \mathrm{GL}_r(\mathbb{C}))/\mathrm{GL}_r \to \mathrm{Ch}^{\pi}_{r,\mathbb{C}}$ is surjective but not injective in general.
(2) $\mathrm{Ch}^{\pi}_{r,\mathbb{C}}$ parameterizes semi-simple isomorphism classes of representations $\pi_1(X, x_0) \to \mathrm{GL}_r(\mathbb{C})$.

A guiding question

Metaquestion

How do 'natural' subvarieties of $\mathrm{Ch}_{r,\mathbb{C}}^{\pi_1(X,x_0)}$ look like if X is an algebraic variety.

Example

For $X = \mathbb{C} \setminus \{0,1\}$, $\pi_1(X,x_0) = \langle g_1, g_2 \,|\, \rangle$, $r = 1$ we get an algebraic torus

$$\mathrm{Ch}_{1,\mathbb{C}}^{\pi_1(X,x_0)} = \mathrm{Hom}(\pi_1(X,x_0), \mathbb{C}^\times) = (\mathbb{C}^\times)^2.$$

A 'nice' quasi-linear subset of this $\mathrm{Ch}_{1,\mathbb{C}}^{\pi_1(X,x_0)}$ is for example $\{(x,y) \in (\mathbb{C}^\times)^2 \,|\, x^j y^k = \mu\}$ ($j,k \in \mathbb{Z}$, $\mu \in \mathbb{C}^\times$ root of unity).

In general for rank 1, $\pi = \pi_1(X,x_0)$

$$\mathrm{Ch}_{1,\mathbb{C}}^{\pi} = \text{(finite group)} \times \text{(algebraic torus)}.$$

Jumping loci I

Cohomology with coefficients in local system $\mathcal{L}$ is finite dimensional $\mathbb{C}$-vector space $H^i(X, \mathcal{L}), h^i(\mathcal{L}) = \dim_{\mathbb{C}} H^i(X, \mathcal{L})$.

Jumping loci

The jumping loci

$$\Sigma^i_X(j) = \{\mathcal{L} \text{ semi-simple} \mid h^i(\mathcal{L}) > j\} \subset \mathrm{Ch}^\pi_{r,\mathbb{C}}$$

are closed subvarieties.

Metaconjecture

If X is an algebraic variety itself then the jumping loci $\Sigma^i_X(j)$ are 'very special' subvarieties of $\mathrm{Ch}^\pi_{r,\mathbb{C}}$ $\rightsquigarrow$ for rank 1 quasi-linear and motivic.

Jumping loci II

Now X is (affine) algebraic variety, e.g. $X = \{x \in \mathbb{C}^N \mid P_i(x) = 0\}$ with $P_i \colon \mathbb{C}^N \to \mathbb{C}$ polynomials.

Theorem (Green-Lazarsfeld, Simpson, Budur-Wang,. . ., Esnault-K)

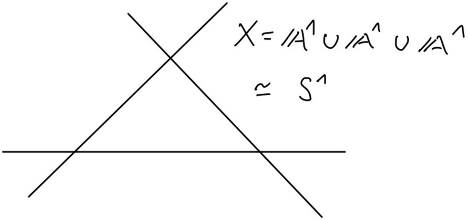
If X is non-combinatorial then for rank 1 the jumping loci $\Sigma^i_X(j)$ are quasi-linear and motivic in $\mathrm{Ch}^{\pi}_{1,\mathbb{C}}$.

Remark

- G-L, S and B-W use complex analysis/geometry and transcendental number theory $\rightsquigarrow$ theorem for $X/\mathbb{C}$ smooth.
- We use arithmetic of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Non-combinatorial varieties

Example: a highly combinatorial variety X:



X non-combinatorial

In general the homotopy type of (affine) algebraic varieties can be anything (of finite type); just glue affine spaces.
We say X is non-combinatorial if for any Zariski dense $U \subset X$ the map $H_1(U, \mathbb{Q}) \to H_1(X, \mathbb{Q})$ is surjective.
Normal varieties are non-combinatorial.

Quasi-linear and motivic

Now $\pi = \pi_1(X, x_0)^{\mathrm{ab}}$ is abelianized fundamental group.

Quasi-linear

$S \subset \mathrm{Ch}^{\pi}_{1,\mathbb{C}} = \mathrm{Hom}(\pi, \mathbb{C}^\times)$ is quasi-linear if it is a finite union of subvarieties of the form

$$\chi \cdot T \text{ with } \chi \in \mathrm{Hom}(\pi, \mathbb{C}^\times) \text{ torsion}, \; T = \mathrm{Hom}(\tilde{\pi}, \mathbb{C}^\times) \text{ subtorus}$$

with $\tilde{\pi}$ torsion free quotient of π.

Motivic

S is motivic if above $\tilde{\pi}$ are quotient Hodge structures of $\pi = H_1(X, \mathbb{Z})$ (for X connected).

Simpson's analytic method

$X/\mathbb{C}$ smooth, projective variety

line bundle L/X, $\nabla\colon L \to L \otimes_{\mathcal{O}_X} \Omega^1_X$ flat connection
$\rightsquigarrow$ rank 1 local system $\mathrm{RH}(L,\nabla) := \ker(\nabla^{\mathrm{an}})$

$\mathrm{Pic}^\nabla(X) =$ variety of flat connections on line bundles$/X$

Riemann-Hilbert correspondence

$\mathrm{RH}\colon \mathrm{Pic}^\nabla(X) \xrightarrow{\sim} \mathrm{Ch}^\pi_{1,\mathbb{C}}$ is an isomorphism of complex manifolds (not of algebraic varieties).

In general $\mathrm{Pic}^\nabla(X)$ is not affine but $\mathrm{Ch}^\pi_{1,\mathbb{C}}$ is affine .

Example

For $X/\mathbb{C}$ smooth, projective curve $\rightsquigarrow$ surjective morphism
$\mathrm{Pic}^\nabla(X) \to J(X)$, $(L,\nabla) \mapsto L$ to projective variety.
$J(X) = (\Omega^1(X))^*/H_1(X,\mathbb{Z}) =$ Jacobian variety

Simpson's theorem (1993)

Assume that X is smooth, projective variety defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$.

Theorem (Simpson)

For a subvariety $S \subset \mathrm{Ch}^{\pi}_{1,\overline{\mathbb{Q}}}$ with $\mathrm{RH}^{-1}(S) \subset \mathrm{Pic}^{\nabla}(X)$ subvariety defined over $\overline{\mathbb{Q}}$
$\Rightarrow S$ is quasi-linear and motivic.

Idea of proof (without torsion property): Assume theorem false and intersect S with suitable algebraic curve $C \subset \mathrm{Pic}^{\nabla}(X)$ like $C = \{(\mathcal{O}_X, d + t\alpha) \,|\, t \in \mathbb{C}\}$ $(\alpha \in \Omega^1(X))$.

- $\mathrm{RH}^{-1}(S) \cap C$ is proper subvariety of C, so is finite.
- $S \cap \mathrm{RH}(C) = \{\text{zeros of } a_1 \exp(u_1 t) + \cdots + a_s \exp(u_s t)\}$ $(a_i \in \mathbb{C}^\times,\ u_i \in \mathbb{C}$ different, $s > 1)$ countable infinite.

New arithmetic method

Idea: Work with $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.

Assume $X/\mathbb{Q}$, $x_0 \in X(\mathbb{Q})$ and let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\widehat{\pi}$ = profinite completion of $\pi = \pi_1(X, x_0)$.

Riemann's existence theorem

There is a natural action of G on $\widehat{\pi}$.

$$\boxed{\varphi\colon \mathrm{Hom}_{\mathrm{cont}}(\widehat{\pi}, \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)) \to \mathrm{Ch}^{\pi}_{\overline{\mathbb{Q}}_\ell}.}$$

Compose $\rho \in \mathrm{Hom}_{\mathrm{cont}}(\widehat{\pi}, \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell))$ with $\pi \to \widehat{\pi}$ and semi-simplify.

Theorem (Esnault-K)

For $r = 1$, $S \subset \mathrm{Ch}^{\pi}_{1,\overline{\mathbb{Q}}_\ell}$ irreducible subvariety, $\varphi^{-1}(S) \neq \varnothing$ stabilized by G
$\Rightarrow$ S is quasi-linear and motivic.

Idea of proof: some ℓ-adic analysis

Using ℓ-adic exponential map

$$\exp\colon z \mapsto \sum_{i=0}^{\infty} \frac{1}{i!} z^i \quad \text{convergent for } |z|_\ell < \ell^{-\frac{1}{\ell-1}} \quad (z \in \overline{\mathbb{Q}}_\ell)$$

one linearizes the problem from $S \subset (\overline{\mathbb{Q}}_\ell^\times)^n$ to

$$S^{\mathrm{Lie}} = \exp^{-1}(S) \subset (\overline{\mathbb{Q}}_\ell)^n.$$

There is a special element $\sigma \in G$ (Bogomolov) such that $\sigma\colon (\overline{\mathbb{Q}}_\ell)^n \to (\overline{\mathbb{Q}}_\ell)^n$ is multiplication by a scalar $\notin$ {roots of unity}.

Exercise from Analysis III (?)

A closed submanifold $\mathcal{S}$ of $(\overline{\mathbb{Q}}_\ell)^n$ stabilized by σ is linear.

Geometric local system

Question

What happens (conjecturally) in the the higher rank case $r > 1$.

Need to consider 'special'(=geometric) points of $\mathrm{Ch}^{\pi}_{r,\mathbb{C}}$.

Example

For $f\colon Y \to X$ a family of smooth projective varieties the vector spaces

$$x \in X \mapsto H^i(f^{-1}(x), \mathbb{C})$$

form a local system.

Such local systems (or slight generalizations) are called geometric.

Density conjecture

Density Conjecture (Simpson/Esnault-K)

$S \subset \mathrm{Ch}^{\pi}_{r,\overline{\mathbb{Q}}_\ell}$ irreducible subvariety, $\varphi^{-1}(S) \neq \varnothing$ stabilized by G
$\Rightarrow$ the geometric points are Zariski dense in S.

Relation to our Theorem ($r = 1$)?

Fact: For $r = 1$ the geometric points are the torsion points of $\mathrm{Ch}^{\pi}_{1,\mathbb{C}}$.

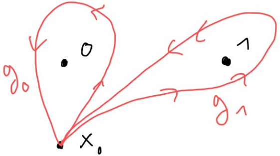
Theorem (Laurent, 1984) aka Manin-Mumford conjecture for tori

Subvariety $S \subset (\mathbb{C}^\times)^n$ such that the torsion points are Zariski dense on S
$\Rightarrow$ S is quasi-linear.

So for $r = 1$:

Density Conjecture $\overset{\text{Thm.(Laurent)}}{\Leftrightarrow}$ quasi-linear part of our theorem.

Weak evidence: quasi-unipotent local monodromy



$X = \mathbb{C} \setminus \{0, 1\}$

loops $g_0, g_1 \in \pi_1(X, x_0) = \pi$

$\rightsquigarrow$ local monodromy

$\rho(g_0), \rho(g_1) \in GL_r(\mathbb{C})$

$\rho \colon \pi \to \mathrm{GL}_r(\mathbb{C})$ representation, $X \subset \overline{X}$ 'nice' compactification.

irreducible component D_i of $\overline{X} \setminus X$ $\rightsquigarrow$ local monodromy operator $\rho(g_i)$ (well-defined up to conjugation).

Monodromy Theorem (Brieskorn, Borel, Grothendieck)

ρ geometric $\Rightarrow$ local monodromy operators $\rho(g_i)$ are quasi-unipotent (eigenvalues are roots of unity).

Weak evidence: quasi-unipotent local monodromy

The density conjecture + Monodromy Theorem together imply the following result.

Theorem (Esnault-K)

$S \subset \mathrm{Ch}^{\pi}_{r,\mathbb{C}}$ irreducible subvariety, $\varphi^{-1}(S) \neq \varnothing$ stabilized by G
$\Rightarrow$ the representations ρ with quasi-unipotent local monodromy $\rho(g_i)$ are Zariski dense.

Further remarks

- Drinfeld (2001) proved that the de Jong conjecture (now a theorem thanks to Gaitsgory *et. al.* based on the geometric Langlands program) implies a weaker form of the Density Conjecture. He used this weak density together the Langlands correspondence for GL_r of function fields (Drinfeld/Lafforgue) to prove Kashiwara's conjecture about cohomology of complex varieties (Hard Lefschetz for semi-simple perverse sheaves etc.).
- An analog of the Density Conjecture for the étale fundamental group over a field $k = \overline{k}$ of positive characteristic would imply that the Kashiwara conjecture (Hard Lefschetz etc.) holds in positive characteristic for ℓ-adic cohomology (one of the big open questions in étale cohomology).
- In characteristic > 0 and for the étale fundamental group the analog of our theorem on quasi-linearity and motivicity holds for rank 1 (Esnault-K).

Thank you very much.